Analytic Pricing of CoCo Bonds*

Colin Turfus
Deutsche Bank
1 Great Winchester St,
London EC2N 2DB

Alexander Shubert
J.P. Morgan
25 Bank Street,
London E14 5JP

November 22, 2016

First Published: June 11, 2015

Abstract

We present a new model for pricing contingent convertible (CoCo) bonds which facilitates the calculation of equity, credit and interest rate risk sensitivities. We assume a lognormal equity process and a Hull-White (normal) short rate process for the conversion intensity with an equity jump on conversion. We are able to derive approximate solutions in closed form which are seen to be remarkably accurate for a wide range of market conditions, even out to long maturities. The method relies on the assumption that the conversion intensity volatility is asymptotically small, but is seen to work adequately even for relatively large conversion intensity volatilities.

Key Words: contingent convertible bond; CoCo bond; jump-diffusion process; closed-form analytic solution; asymptotic expansion; perturbation analysis; equity-credit hybrid.

1 Introduction

A Contingent Convertible (CoCo) bond is a debt instrument that converts into common equity when the issuing financial institution's capital ratio (core Tier 1 capital to risk-weighted assets) falls below a predetermined level or a similar "capital event" occurs, for example a regulator may mandate a conversion event. CoCo bonds are becoming increasingly important as a means of financial institutions obtaining new funding while at the same time preserving the Core Tier 1 capital ratios required by Basel II and Basel III regulation. CoCo bonds and the various methods which have been proposed for their pricing have recently been reviewed by Wilkens and Bethke (2014). They distinguish three categories of model and give several examples of each.[1]

- *Structural models* attempt to capture the trigger event by modelling the core Tier 1 capital level (and risk-weighted assets) directly; see e.g. Pennacchi (2010) and Brigo et al. (2015).
- *Equity derivatives models* instead model the equity price, taking this as a proxy for the financial health of the company and hence of its Tier 1 capital ratio. This modelling approach was first expounded by De Spiegeleer and Schoutens (2012a).
- *Credit derivatives models* rather model an equity conversion intensity process analogous to (or possibly in addition to) a credit default intensity process. This modelling approach was first expounded by Serjantov (2011) and by De Spiegeleer and Schoutens (2012a).

*The views expressed herein should not be considered as investment advice or promotion. They represent personal research of the authors and do not purport to reflect the views of their employers (current or past), or the associates or affiliates thereof.

[1] The reader is referred to the review by Wilkens and Bethke (2014) for a more extensive survey of the literature on the subject of CoCo bond pricing than we have sought to provide here.

A clear advantage of the former (structural) approach is that what is modelled is closer to the contractual definition of the payoff. However the parametrization of a stochastic model based on the unobservable details of a financial institution's fluctuating assets is problematic, both in theory and in practice, and renders the price determination process opaque to those used to more traditional bond pricing methodologies. Further, it is not obvious how to calculate sensitivities of model prices to "market parameters" when calibration involves the use of balance-sheet figures published only infrequently.

While the latter two methods overcome these challenges to some extent, a shortcoming they share is their need for the payoff at default to be specified as an *input* of the model (albeit calibrated to the market), rather than as an output. Thus, although the equity derivatives approach allows equity delta values to be calculated, it is questionable whether these can be entirely believed when the payoff is contractually determined to be proportional to the equity price (albeit capped) but the model takes this to be a predetermined parameter. Further, the equity derivatives approach does not allow conversion risk (CS01) to be calculated at all, since there is no concept of credit spread in the model. This is likely to be seen as a significant drawback by the many bond traders who habitually think of bond pricing in terms of credit spreads with associated sensitivities and who would be reluctant to depart too far from a methodology where the model inputs have a transparent relationship with the output prices. Similarly, while a credit derivatives approach would allow for calculation of CS01, it has no concept of an equity spot process, so is not capable of quantifying equity risk.

We see the new approach which we develop below as an advance on what has been proposed thus far in the literature, capturing as it does and building on the benefits of the latter two of the above approaches. It is in essence a hybrid credit-equity derivatives model wherein *both* the equity price and the conversion intensity process are modelled as (correlated) stochastic processes. A possible downward jump of the equity price is allowed to occur consequent upon a conversion event, something the market appears to price in.[2] Despite its greater complexity, this hybrid model turns out to be almost as tractable as the single-factor models (albeit that a certain level of approximation needs to be introduced associated with the asymptotic analysis involved). Two immediate advantages are that the equity price at conversion is *modelled* rather than being provided as a model input; and both equity risk *and* credit risk sensitivity are amenable to separate calculation, thus facilitating hedging and/or VaR reporting.

We present our mathematical model as a PDE which is amenable to numerical solution by a finite difference approach. However we describe here an alternative approach based on the method of asymptotic expansions which gives rise to closed-form solutions of good accuracy. The approximation method used parallels that propounded by Kim and Kunitomo (1999) to take account of the impact of stochastic rates on vanilla equity option prices. A similar perturbation expansion approach has also been employed successfully by Fouque et al. (2011) to capture the impact of stochastic volatility on a range of options including binary, barrier, Asian and American options on equity underlying, bond options, defaultable bonds and single- and multi-name credit derivatives such as NTDs and CDOs, using their two time-scale stochastic volatility approach. Credit derivatives pricing was also considered by Muroi (2005) and Muroi and Takino (2010). Related work has been done by Alòs (2006) and Antonelli and Scarlatti (2009) to take account of the impact of stochastic volatility on equity options and by Benhamou et al. (2010) in the context of a local volatility model. The perturbation approach has been further extended in the context of stochastic volatility to address exchange options by Antonelli et al. (2010), quanto options by Park et al. (2013), Asian options on oil futures contracts by Shiraya and Takahashi (2011), multi-asset cross-currency options by Shiraya and Takahashi (2015), barrier options by Shiraya et al. (2011) and swaptions by Shiraya et al. (2012), and in the context of local volatility to address best-of options on equity and inflation by Gobet and Hok (2014). To our knowledge ours is the first attempt to apply such methods to a hybrid credit-equity model.

The founding assumptions of the mathematical model are set out in §2 below. The value of the survival-contingent cash flows is derived in §3.1 and the calculation of the equity recovery value of the CoCo bond is set out in §3.2, the main result of which is Eq. (27). The extension of the scope of the modelling approach to

[2] The model is in fact mathematically equivalent to that proposed by Ehlers and Schönbucher (2006) and independently by EL-Mohammadi (2009) in relation to Quanto CDS pricing, with the exchange rate from the foreign to the domestic currency taking the role of the stock price in our model.

address perpetual CoCo bonds and CoCo bonds for which the conversion price is floating rather than fixed is considered in §3.3. The calibration of the model to market prices is considered in §4. The accuracy of the asymptotic result is considered in §5 in comparison with results obtained by direct finite difference solution of the associated PDE. Conclusions and proposals for further research are presented in §6

2 Formal derivation of model

Our starting point for CoCo bond pricing is a stochastic model involving an equity conversion intensity and an equity price. The evolution of the former is assumed to be governed by a diffusive process, and in the latter case by a jump-diffusion process, with a downward jump of a fixed relative amount occurring at the time of the conversion event. The diffusive processes are assumed to be (negatively) correlated.

We follow the CoCo bond pricing approach of previous authors in taking interest rates to be deterministic. The appropriateness of this assumption we revisit in §5 below. We take the instantaneous forward rate to be given by a bounded L^1 function $\overline{r}(t)$ defined on $[t_0, T_m]$, with t_0 the observation date and $T_m < \infty$ a future date chosen to be greater than the maturity of any bond to be priced under the calibrated model. We do not model a default process as such for the debt-issuing bank. Rather we take the position that default and conversion events are 100% correlated (or close thereto), whence, assuming the standard default copula model of Li (2000), the event with the greatest intensity can be taken always to occur first. Since in practice this will invariably be the conversion intensity, we can for CoCo bond pricing purposes effectively ignore default events.

The essential detail of our model is that the equity process is given by Eq. (5) and the equity conversion process by Eqs. (9)-(11), with the correlation between the processes given by Eq. (1). The reader less interested in the mathematical formalism may, based on this understanding, wish to proceed directly to the next section.

Formally we will work with a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^j$ where the Brownian filtration $\{\mathcal{F}_t^W : t \in [t_0, T_m]\}$ is that generated by a two-dimensional (correlated) Brownian motion $W = (W_\lambda, W_S)$ with

$$\mathrm{corr}(W_{S,t}, W_{\lambda,t}) = \rho_{\lambda S} \tag{1}$$

and the filtration $\{\mathcal{F}_t^J : t \in [t_0, T_m]\}$ is that generated by a Cox process n_t whose intensity λ_t is taken to depend upon W_λ. We take the first jump in this Cox process to represent a conversion event and denote the associated stopping time by $\tau > t_0$. We shall henceforth refer to λ_t as the *equity conversion intensity* process. We shall require in practice that n_t and λ_t be defined only for $t \in D_m := [t_0, \min\{\tau, T_m\}]$.

We adopt the hypothesis $\mathbb{H}$ of Ehlers and Schönbucher (2006) concerning the equivalence of $\mathcal{F}$- and $\mathcal{F}^W$-martingales, which hypothesis has the following corollary:

Assumption 2.1 *For any $\mathcal{F}^W$-adapted g satisfying $E[\int_{t_0}^{T_m} |g_s| ds] < \infty$, we have*

$$E\left[\int_t^T \mathbb{1}_{\{s<\tau\}} g_s ds \middle| \mathcal{F}_t\right] = \mathbb{1}_{\{t<\tau\}} \int_t^T E\left[e^{-\int_t^s \lambda_u du} g_s \middle| \mathcal{F}_t\right] ds.$$

The equity conversion intensity process λ_t is taken to be governed on D_m by

$$d\lambda_t = m_\lambda(\lambda_t, t)dt + \sigma_\lambda(\lambda_t, t)dW_{\lambda,t}. \tag{2}$$

For reasons of tractability, we will look to use a Hull-White model for the conversion intensity, whence we specify that its volatility depends only on t, viz.

$$\sigma_\lambda(\lambda_t, t) \equiv \sigma_\lambda(t).$$

We further assume that $\sigma_\lambda(.)$ is a bounded L^2 function on D_m.

The equity process is defined on D_m by

$$dS_t = \mu_S(S_t, t)dt + S_t \sigma_S(t) dW_{S,t} + S_t^- k(dn_t - \lambda_t dt). \tag{3}$$

with $\sigma_S(.)$ likewise a bounded L^2 function on D_m. This formulation gives rise to an equity price jump of size k with $-1 < k < 0$, contingent on an equity conversion event.

To proceed, we adopt a pricing measure $\mathbb{Q}$ defined with respect to the money market numéraire

$$\beta(t) := \exp\left(\int_{t_0}^{t} \overline{r}(s)ds\right). \tag{4}$$

By an argument analogous to that set out by Ehlers and Schönbucher (2006) (see in particular their §4.1), we conclude that under $\mathbb{Q}$ Eq. (3) becomes

$$dS_t = (\overline{r}(t) - \delta(t) - k\lambda_t)S_t dt + \sigma_S(t)S_t dW_{S,t} + S_t^- k dn_t, \tag{5}$$

where $\delta(t) \geq 0$ is a supposed deterministic dividend rate for the equity (taken to be a bounded L^1 function on D_m) and

$$S_t^- := \lim_{u \to t^-} S_u$$

to ensure the correct behaviour at stopping times τ. They argue that Eq. (5) has a unique positive strong solution if λ_t is progressively measurable with respect to the filtration $\mathcal{F}$, which under our proposed use of a Hull-White model and their hypothesis $\mathbb{H}$ will be the case.

Turning then back to Eq. (2), the adoption of a Hull-White model requires that we take

$$m_\lambda(\lambda_t, t) = a(\tilde{\theta}(t) - \lambda_t) \tag{6}$$

for some deterministic function $\tilde{\theta}(t)$ to be determined by a no-arbitrage condition. To this end we posit the existence of a market of (conversion-)risky bonds paying unit notional at time $t_i > t_0$ if no conversion has taken place and zero otherwise, for $i = 1, 2, \ldots, N_B$, say. Applying our pricing measure $\mathbb{Q}$, we see the price of such bonds can be expressed as[3]

$$\begin{aligned} B_0(t_i) &:= E\left[\mathbb{1}_{\{t_i < \tau\}} \frac{\beta(t_0)}{\beta(t_i)}\right] \\ &= e^{-\int_{t_0}^{t_i} \overline{r}(s)ds} E\left[e^{-\int_{t_0}^{t_i} \lambda_s ds}\right], \end{aligned}$$

The availability of such prices allows us to define a forward conversion intensity function $\overline{\lambda}(t)$, $t \in D_m$, consistent with the stipulation that

$$B_0(t) = B(t_0, t) \tag{7}$$

where we define

$$B(u, v) := e^{-\int_u^v (\overline{r}(s) + \overline{\lambda}(s))ds}. \tag{8}$$

The required functional form of $\overline{\lambda}(t)$ can then be obtained from Eq. (7) and the market prices $B_0(t_i)$ by a standard bootstrapping procedure. The function $\overline{\lambda}(t)$ will by construction typically be a piecewise-linear continuous function. But we shall require formally only that it be piecewise differentiable and bounded on $[t_0, T_m]$. We immediately infer that the risk-neutral survival probability under $\mathbb{Q}$ is given by

$$\begin{aligned} Q(t) &:= \mathbb{Q}(\tau > t) \\ &= e^{-\int_{t_0}^{t} \overline{\lambda}(s))ds}. \end{aligned}$$

Applying the methodology set out by Hull and White (1990) to our intensity short rate model, we find that under $\mathbb{Q}$ Eq. 2 becomes

$$d\lambda_t = -a(\lambda_t - \theta(t))dt + \sigma_\lambda(t)dW_{\lambda,t} \tag{9}$$

[3] We make use here of Assumption 2.1 in the limiting case where g is a Dirac δ function.

for some deterministic function $\theta(t)$. This is easily solved by direct integration under the restrictions we have placed on the drift and the diffusion terms. Replication of the (known) term structure of survival probabilities requires that we choose

$$\theta(t) = \frac{1}{a}\left(\frac{d\overline{\lambda}(t)}{dt} + a\overline{\lambda}(t) + I_\lambda(t_0, t)\right), \tag{10}$$

with[4]

$$I_\lambda(t_1, t_2) := \int_{t_1}^{t_2} e^{-2a(t_2-u)}\sigma_\lambda^2(u)du. \tag{11}$$

The solution for λ_t is adapted to the filtration $\mathcal{F}^W$ on D_m and hence also to $\mathcal{F}$.

3 CoCo bond pricing formulae

To proceed further, we follow Ayache et al. (2002) who proposed in pricing convertible bonds that the survival-contingent flows be dealt with separately from the conversion-contingent flows, expressing the PV of a CoCo bond with notional N as

$$V(t) = Nf_{\mathrm{surv}}(t) + Nf_{\mathrm{conv}}(t). \tag{12}$$

Here the first term on the r.h.s. is the part of the PV generated by survival-contingent cash flows, i.e. the coupon payments and notional repayment. The second term is the contribution to the PV resulting from cash flows contingent on a conversion event at time $\tau < T$, the bond maturity. We will consider these in turn.

3.1 Valuation of the survival-contingent cash flows

The value at time t_0 of the cash flows contingent on no conversion event for a bond paying a fixed coupon c can be conveniently expressed as

$$f_{\mathrm{surv}}(t_0) = \sum_{i=1}^{n} c\Delta t_i B_0(t_i)\mathbb{1}_{t_i>t_0} + B_0(T) \tag{13}$$

where $\{t_1, t_2, \ldots, t_n = T\}$ are the coupon payment dates, Δt_i the relevant day count fractions and $B_0(t)$ is as defined in Eq. (7) above. Similarly for a floater paying a fixed spread s over Libor,

$$f_{\mathrm{surv}}(t_0) = \sum_{i=1}^{n} (F_i(t_0) + s)\,\Delta t_i B_0(t_i)\mathbb{1}_{t_i>t_0} + B_0(T) \tag{14}$$

where $F_i(t_0)$ is the relevant forward Libor rate for the payment period ending at time t_i. Note that for a perpetual bond, we can conveniently let $n \to \infty$ and remove the notional repayment term $B_0(T)$ in the above expressions. It is a straightforward matter to see, under some mild assumptions, that the resultant infinite series are convergent (see Corollary 3.1).

3.2 Estimation of the equity recovery value

We next analyse the payoff made in the event of conversion and derive a pricing equation in PDE form to determine its value. Following that, we seek a closed-form solution to the pricing equation using a perturbation expansion approach.

[4] Although we have assumed that $a > 0$ is a constant, we could equally have taken $a(.)$ to be a real-valued L^1 function of t. The analysis goes through identically with the exception that, where $e^{-a(v-u)}$ appears in formulae below, this should be replaced by $e^{-\int_u^v a(t)dt}$.

3.2.1 Equity recovery payoff

We suppose that the evolution of $f_{\text{conv}}(t)$ is captured by a stochastic process $\hat{h}(S_t, \lambda_t, t)$. Our interest is in calculating $\hat{h}(S_{t_0}, \overline{\lambda}(t_0), t_0)$. Our main conclusion in this section is that this can be approximated to a good level of accuracy by the closed form expression in Eq. (27).

We propose as our model for the payoff of the CoCo bond upon conversion at $t = \tau$ that this is given by $\hat{h}_{\text{conv}}(S_\tau^-)$ where

$$\hat{h}_{\text{conv}}(S) := \min\{MS(1+k), K\} \tag{15}$$

with M the number of shares issued per unit notional on conversion, and $K \leq 1$ specifying a cap on the value of the equity payoff.

We deduce from the above that $f_{\text{conv}}(t)$ is given by

$$f_{\text{conv}}(t) = E\left[\mathbb{1}_{t<\tau<T} \min\{MS_\tau^-(1+k), K\} \frac{\beta(t_0)}{\beta(\tau)} \middle| \mathcal{F}_t\right] \tag{16}$$

for $t \in [t_0, T]$. Given that the payoff is a continuous bounded function of the underlying S_t and is furthermore adapted to the filtration, the above expectation is well-defined and finite.

3.2.2 Interpretation of payoff specification

A few words should be said by way of interpretation about the form specified for the payoff here, in particular the cap. There are several reasons for imposing this. In the first instance, the contract may specify such a cap. However, even where no cap is specified contractually, it is clearly unrealistic to expect that the bond holder would ever receive an equity payoff in excess of the face value of the bond, even though the model theoretically admits of this possibility: the issuing bank would address any Tier 1 capital shortfall by issuing new equity on the market rather than converting CoCo bonds. On that basis, a value of K at most equal to 1 should be used in all cases. Also, as we shall see in §3.3.2 below, for CoCo bonds which are specified to convert to equity at the market price S_τ, the market price is usually floored leaving the payoff profile looking the same as for the case considered here, which means that our methodology can be conveniently adapted to cover that case too.

A word of clarification should also be offered about the role of the jump size k in the payoff. Although in our idealised model this occurs exactly at the stopping time, the reality is that the jump associated with conversion will not be instantaneous but rather will occur over a time period. While there will certainly be a jump after the conversion event associated with dilution of the pool of shares, we shall see in §4 below that the jump sizes needed to fit market prices tend to be larger than could be considered explicable in terms of a dilution effect. So the jump k should really be interpreted as a market phenomenon, likely occurring partly before and partly after the conversion event. In that sense, it should be thought of more realistically as an accumulation of non-diffusive downward movements rather than a pure jump. Until such time as a conversion event becomes imminent, hoever, the difference is largely immaterial from a modelling perspective. But some care needs to be taken.

In particular, in a distressed situation where the stock price was debased and a conversion event looked imminent, one would have to consider whether part of the expected jump k had already occurred and was being priced into the CoCo bonds and reduce the value of k used in the model to a less negative one. This might appear to be an unwelcome arbitrariness in model calibration/parametrisation but, as we shall suggest in §3.3.2 and §4 below, should not be viewed as a significant drawback.

3.2.3 Perturbation Analysis

We next look to derive a closed-form solution for $f_{\text{conv}}(t_0)$ by means of a perturbation expansion approach. To this end we first reformulate the problem as a PDE, deducing by application of the well-known Feynman-Kac method that, under the assumptions set out in §2, $\hat{h}(S_t, \lambda_t, t)$ will be governed by the following backward diffusion equation:

$$\hat{\mathcal{L}}[\hat{h}(S, \lambda, t)] = -\lambda \hat{h}_{\text{conv}}(S), \tag{17}$$

where

$$\hat{\mathcal{L}}[.] := \frac{\partial}{\partial t} + (\overline{r}(t) - \delta(t) - k\lambda)S\frac{\partial}{\partial S} + a(\theta(t) - \lambda)\frac{\partial}{\partial \lambda}$$
$$+ \frac{1}{2}\left(\sigma_S^2(t)S^2\frac{\partial^2}{\partial S^2} + 2\rho_{\lambda S}\sigma_\lambda(t)\sigma_S(t)S\frac{\partial^2}{\partial S\partial\lambda} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial\lambda^2}\right) - (\overline{r}(t) + \lambda) \tag{18}$$

for $t \in D_m$ subject to a final condition $\hat{h}(S_T, \lambda_T, T) = 0$.[5]

The basis of our approximation will be the assumption that $\sigma_\lambda(t) = O(\epsilon)$ where ϵ is an asymptotically small parameter to be defined. We also thereby expect $\lambda_t = \overline{\lambda}(t) + O(\epsilon)$. To ensure a consistent limit in Eq. (17) as $\epsilon \to 0$, we should define ϵ in non-dimensional terms. Since all terms in that equation remain $O(1)$ in that limit we focus on Eq. (10) and are led to choose a definition which captures the ratio between the second and third terms on the r.h.s.. Averaging over the life of the trade we therefore propose

$$\epsilon^2 := \frac{\int_{t_0}^T \sigma_\lambda^2(t)dt}{a^2\int_{t_0}^T \overline{\lambda}(t)dt}.$$

For reasons of tractability we would like also to remove the first-order derivative terms in Eq. (17), at least to leading order. To that end we formally propose a scaled characteristic stochastic intensity coordinate y_t defined by

$$y_t := \epsilon^{-1}(\lambda_t - \overline{\lambda}(t))e^{a(t-t_0)} \tag{19}$$

and a scaled volatility parameter $\sigma_y(t)$ defined by

$$\sigma_y(t) := \epsilon^{-1}\sigma_\lambda(t)e^{a(t-t_0)}.$$

We further define an equity term variance

$$I_S(t_1, t_2) := \int_{t_1}^{t_2} \sigma_S^2(u)du, \tag{20}$$

a new characteristic stochastic equity price coordinate x_t such that

$$S_t = F(t)e^{x_t - \frac{1}{2}I_S(t_0,t)}, \tag{21}$$

where

$$F(t) = S_{t_0}e^{\int_{t_0}^t(\overline{r}(s)-\delta(s)-k\overline{\lambda}(s))ds}, \tag{22}$$

and a payoff function

$$M_0(x,t) := M(1+k)F(t)e^{x-\frac{1}{2}I_S(t_0,t)}.$$

Changing variable from S_t to x_t and from λ_t to y_t and re-expressing $f_{\text{conv}}(t) = h(x_t, y_t, t)$, Eq. (17) is replaced by

$$\mathcal{L}[h(x,y,t)] = -\left(\overline{\lambda}(t) + \epsilon y e^{-a(t-t_0)}\right)\min\{M_0(x,t), K\} + \epsilon y e^{-a(t-t_0)}\left(h + k\frac{\partial h}{\partial x}\right)$$
$$- \epsilon e^{-a(t-t_0)}I_y(t_0,t)\frac{\partial h}{\partial y} \tag{23}$$

[5] A result analogous to Eq. (17) is derived by Ayache et al. (2002) in section 10 of their paper on convertible bond pricing, the main differences being that a) rather than a conversion intensity, they work with a *default* intensity, which they furthermore assume to be deterministic, and b) they apply a different final condition at $t = T$ which allows conversion to be optionally made at the discretion of the bond holder.

for $t \in D_m$ with final condition that $h(x,y,T)=0$, where $\mathcal{L}[.]$ is a standard forced diffusion operator given by

$$\mathcal{L}[.] := \frac{\partial}{\partial t} + \frac{1}{2}\left(\sigma_S^2(t)\frac{\partial^2}{\partial x^2} + 2\rho_{\lambda S}\sigma_S(t)\sigma_y(t)\frac{\partial^2}{\partial x \partial y} + \sigma_y^2(t)\frac{\partial^2}{\partial y^2}\right) - (\overline{r}(t) + \overline{\lambda}(t)) \tag{24}$$

and

$$I_y(t_1,t_2) = \int_{t_1}^{t_2} \sigma_y^2(u)du. \tag{25}$$

As we have already seen in §3.2.1, a bounded solution exists for $h(x,y,t)$ for $t \in D_m$. In the absence of a closed form solution, we pose an asymptotic expansion

$$h(x,y,t) = h_0(x,t) + \epsilon h_1(x,y,t) + \epsilon^2 h_2(x,y,t) + O(\epsilon^3) \tag{26}$$

and solve for successive orders in ϵ iteratively. The method of calculation is fairly standard (see Appendix A). Substituting the results obtained for the first two terms in Eq. (26), setting $t = t_0$ and reverting to unscaled notation, we obtain the following first order accurate result:

Theorem 3.1 *The value of the equity recovery payoff on a CoCo bond can be estimated under our modelling assumptions as follows:*

$$\begin{aligned} f_{conv}(t_0) = {} & M(1+k)\int_{t_0}^{T} B(t_0,v)F(v)\left(\overline{\lambda}(v) + I_R(t_0,v) - \overline{\lambda}(v)\int_{t_0}^{v} I_R(t_0,u)\,du\,D_1\right) N(-d_1(x,t_0,v))\Big|_{x=0}\,dv \\ & + K\int_{t_0}^{T} B(t_0,v)\overline{\lambda}(v)N(d_2(0,t_0,v)))\,dv + O(\epsilon^2) \end{aligned} \tag{27}$$

where

$$\begin{aligned} I_R(t_1,t_2) &:= \rho_{\lambda S}\int_{t_1}^{t_2} e^{-a(t_2-u)}\sigma_\lambda(u)\sigma_S(u)du, && (28)\\ d_2(x,t,v) &:= \frac{\ln M_0(x,v) - \ln K}{\sqrt{I_S(t,v)}}, && (29)\\ d_1(x,t,v) &:= d_2(x,t,v) + \sqrt{I_S(t,v)}, && (30)\\ N(x) &:= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du, && (31)\\ D_1 &:= 1 + k + k\frac{\partial}{\partial x}. && (32) \end{aligned}$$

Proof 3.1 *For the proof of Theorem 3.1, see Appendix A.*

Eq. (27), together with Eqs. (12) and (13) or (14), gives our result for the CoCo bond PV. By way of explanation, the contributions from the first term in parentheses in the first integral and the second integral on the r.h.s. of Eq. (27) represent the value of the capped equity recovery payment subject to an assumed *deterministic* conversion intensity. The second term represents the direct impact of correlation between stochastic fluctuations in the intensity of conversion events and those associated with the recovery amount; since the correlation is always assumed negative, the result is a *decrease* in the PV. The third term captures indirect correlation effects associated with a) non-deterministic risky discounting of the equity recovery flows and b) non-deterministic jump-compensating drift in the equity; their combined effect is such as to diminish the impact of the previous effect.

3.3 Extensions of the main result

3.3.1 Perpetual bonds

It is not common that CoCo bonds are defined to be perpetual, especially when they are callable. This situation can conveniently be handled by taking the bond price to be given by the limit price of a series of bonds with maturity $T^{(i)}$ for increasing $T^{(i)}$. In this case we allow the time T_m up to which the model is taken to be valid to be $T^{(i)}$ in successive calculations. It needs only to be demonstrated that the resultant series $V^{(i)}(t_0)$ is indeed convergent.

Corollary 3.1 *The bond pricing formulae given by Eqs. (12) and (27) together with Eq. (13) or Eq. (14) remain valid in the limit as $T \to \infty$ subject to the following condition being satisfied:*

- $\exists i^* \in \mathbb{N}$ *and* $\zeta_m, \zeta_M \in \mathbb{R}$ *s.t.* $\forall t > t_{i^*}$ *we have:*
  - *a)* $\overline{r}(t) + \overline{\lambda}(t) > \zeta_m > 0$;
  - *b)* $0 < \overline{r}(t) < \zeta_M$.

Proof 3.2 *For the proof of Corollary 3.1, see Appendix B.*

3.3.2 Equity conversion at market price

For many CoCo bonds, rather than the conversion price being fixed in advance contractually, as has been assumed above by taking M to be a fixed value, conversion takes place at the market price on the day the conversion event is announced (see, e.g. De Spiegeleer and Schoutens (2012a)). In such cases, there is still expected to be a downward jump in the value of the equity actually received as a result of dilution following the conversion event. But other than that the payment is effectively a fixed amount. If the size of the jump is taken into account (at an assumed known level), this case can be priced by treating it as a standard risky bond with the credit default intensity replaced by a (larger) conversion intensity.

However, it is common that the conversion price is also floored and given by, say, $S^* = \max\{S_\tau, \overline{F}\}$. The effective payoff per unit notional in the event of conversion will in this case be

$$\begin{aligned} \hat{h}_{\text{conv}}(S_\tau) &= \frac{S_\tau}{S^*}(1+k) \\ &= (1+k)\min\left\{\frac{S_\tau}{\overline{F}}, 1\right\} \end{aligned} \tag{33}$$

But this payoff structure is exactly analogous to Eq. (15), with $M = 1/\overline{F}$ and $K = 1 + k$. The previous results can therefore be used equally to handle this case.

If, as suggested in §3.2 above, some of the "jump" is considered to occur prior to the conversion event, a view will need to be taken about how the market equity price is inferred from the pre-jump *modelled* equity price S_τ. If we assess that a CoCo bond conversion event will trigger a subsequent downward jump of k_{sub} on the equity price with $0 > k_{sub} > k$, we would infer a prior jump of k_{prior} such that

$$1 + k_{prior} = \frac{1+k}{1+k_{sub}},$$

and thus a "market price" of $S_t(1 + k_{prior})$. The payoff should on this basis be taken to be

$$\hat{h}_{\text{conv}}(S_\tau) = (1 + k_{sub})\min\left\{S_\tau \frac{(1+k_{prior})}{\overline{F}}, 1\right\}$$

If, as is sometimes the case, an *average* market price over a period leading up to the conversion event is contractually specified, further adjustment would need to be made to accommodate this.

4 Fitting to market prices

Before presenting results obtained with Eq. (27), we illustrate how the model can be configured so that generated prices fit the market. We consider a selection of six Lloyds CoCo bonds issued in December 2009 for which market data were available. These bonds were all issued with a capital ratio trigger of 5% and a contractual equity exchange price of 0.5921 GBP, corresponding to $M = 1.6889$. Market data from March 5th, 2015 were used in setting up a finite difference solver for Eq. (17). The equity price stood at 0.81 GBP. The conversion intensity volatility was taken to be $\sigma_\lambda = 3\%$. This was chosen on the basis that implied lognormal ATM volatilities for default swaptions are found usually not to exceed about 70%. With, as we shall see, conversion intensities calibrating as low as 4.5%, the value of $\sigma_\lambda = 3\%$ seems reasonable, although arguably this value could have been increased in the cases where the calibration produced higher conversion intensity values. The mean reversion rate was taken to be $a = 0.35$, following Brigo and Mercurio (2006) who report this as the calibrated mean reversion rate they obtained for their Shifted Square Root Diffusion (SSRD) credit default intensity model. We also considered a value of $a = 0.25$ but this did not impact on any calibrated intensity values by more than 10bp, leading us to conclude the precise mean reversion value chosen is not crucial. For convenience the conversion intensity was taken to be a constant, λ, although in a more careful calibration a default intensity might be inferred from the term structure of CDS rates and a constant "CoCo" spread added to obtain $\overline{\lambda}(t)$. A value of $\rho_{\lambda S} = -0.6$ was chosen on the basis of evidence gleaned from time series analysis for financial institutions; see, for example, Byström (2005). Various values of k were considered between -0.6 and -0.9. The impact of this choice and the chosen value of λ on the PV of two of the bonds considered is shown in Figs. 1 and 2 in relation to the market price.

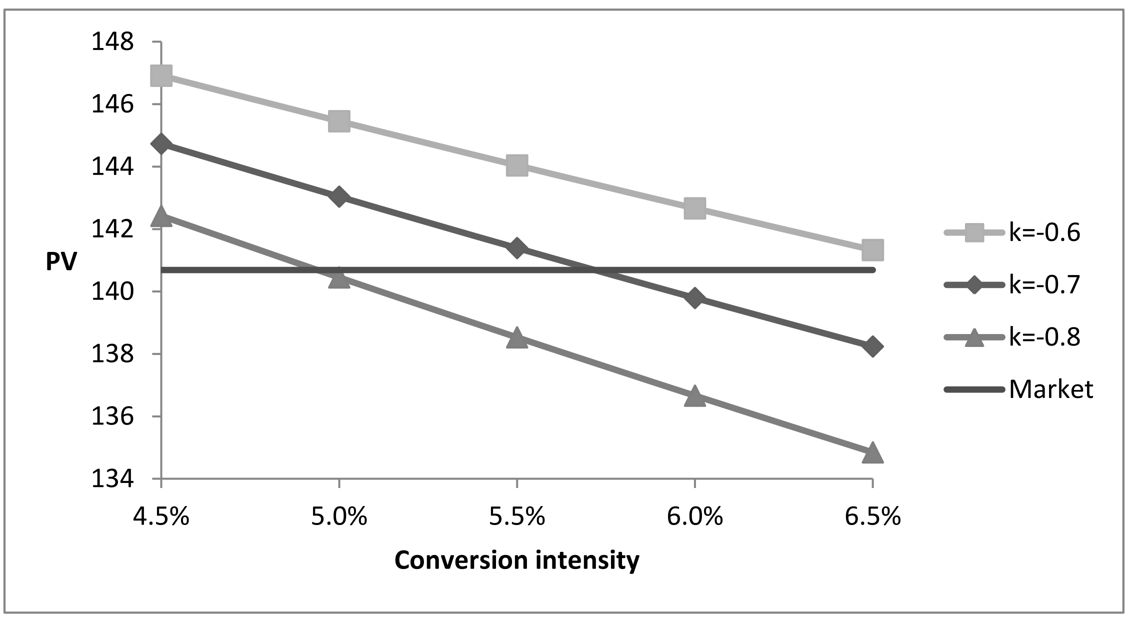

Figure 1: Impact of conversion intensity and jump magnitude on PV of Lloyds CoCo bond with ISIN XS0459089255.

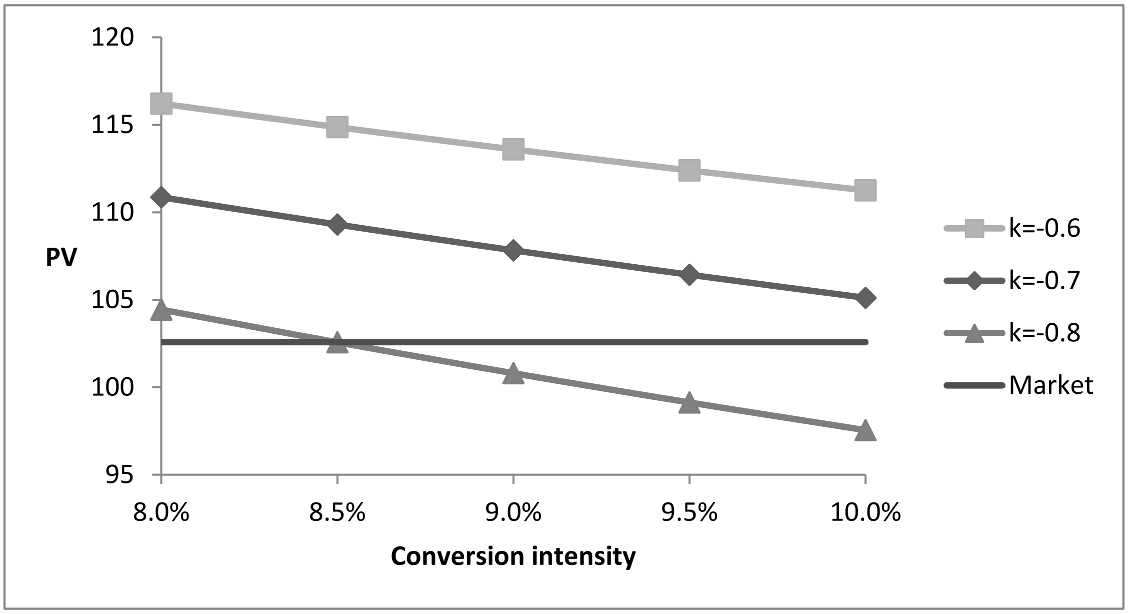

Figure 2: Impact of conversion intensity and jump magnitude on PV of Lloyds CoCo bond with ISIN XS0459086822.

ISIN	Coupon	Maturity	Price	k	λ
XS0459089255	15%	21-Dec-2019	140.69	-0.8	4.9%
XS0459090774	7.375%	12-Mar-2020	106.57	-0.8	6.1%
XS0459088794	6.385%	12-May-2020	107.00	-0.8	4.5%
XS0459092473	10.5%	29-Sep-2023	103.49	-0.9	9.7%
XS0459092556	11.875%	01-Sep-2024	102.81	-0.9	11.7%
XS0459086822	7.975%	15-Sep-2024	102.58	-0.8	8.5%

Table 1: Calibration of Lloyds CoCo bonds to market prices, March 5th, 2015

The details of all the bonds considered and the results of the calibration process are displayed in Table 1. As can be seen, the longer maturity bonds generally require a greater conversion intensity to fit the market price, although there was insufficient consistency between prices of bonds with similar maturities for any formal term structure of conversion intensity to be inferred. On the evidence, the choice of $k = -0.8$ worked reasonably well for four of the six bonds considered. For the other two, a value of $k = -0.9$ was preferred to avoid the need for unduly high λ values.

Clearly these levels of k are much higher than can be attributed to a dilution effect alone, for the reasons discussed in §3.2.2 above. In support of the appropriateness of the levels chosen, we note that, if the implied λ values are used as the default intensity for an equivalent *fixed recovery* bond, the CoCo bond market price is reproduced in all cases considered for fixed recovery values between 0.185 and 0.32, which seems a reasonable level for a subordinated bond.[6] Choosing a smaller jump size leads to higher values of λ being needed and results in higher equivalent fixed recovery levels, which appears undesirable. It could be argued that the CoCo bond model should be calibrated with *larger* jump magnitudes and correspondingly *lower* effective recovery rates. Although this flexibility about multiple (λ, k) pairs being capable of fitting the market price might seem an unwelcome feature of the model, the situation is analogous to the postulation

[6] De Spiegeleer and Schoutens (2012b) using their equity derivatives approach infer an equity recovery level of 19.6% for XS0459089255 based on a debased Lloyds stock price of 0.47 GBP on June 10th, 2011, which level would appear to be commensurate with the sort of recovery levels we propose here.

of a recovery level in vanilla bond pricing; except in that we have past experience to guide our estimation of a "reasonable" recovery level in that case, whereas no CoCo bonds have yet triggered giving us no real evidential basis upon which to decide the matter. The difficulty may also be compounded by the possible impact of the existence of other CoCo bonds with different triggers on the conversion process for any given CoCo bond, as suggested by De Spiegeleer and Schoutens (2012b).

We would further mention that, in the expression for $f_{\text{conv}}(t_0)$ above, $\overline{\lambda}(.)$ and $\sigma_\lambda(.)$ appear almost always in a product with $1+k$, so there is not as much flexibility as might appear. Indeed the effective difference between (λ, k) pairs is, as discussed above, mainly in the relative importance of the recovery payment *vis a vis* the other contractual cash flows. Furthermore, $\rho_{\lambda S}$ and $\sigma_\lambda(.)$ only appear as a product—through $I_R(.)$—in the expression for $f_{\text{conv}}(t_0)$ and nowhere else, so again the flexibility afforded by multiple parameters and the dependency thereon of the results obtained are less than might first appear.

It might further be argued that k more properly ought to be modelled as a random variable. However, the result of this would effectively be equivalent to using a value of k equal to the mean of the assumed distribution, hence not a feature considered worth incorporating into our model.

5 Results and comparative analysis

Computations were made of the equity recovery value of a range of CoCo bonds based on the first line of Eq. (27) (zero-order expansion) and on the full first-order expansion. We chose values of $\lambda = 6\%$, $\sigma_S = 30\%$, $\sigma_\lambda = 3\%$, $a = 35\%$, $\rho_{\lambda S} = -0.6$ and $k = -0.8$. The notional N was taken to be 100; the initial share price S_{t_0}, the share issue rate M and the cap K were all taken to be 1. The results are compared in Fig. 3 and Table 2 with direct finite difference solutions of Eq. (17).

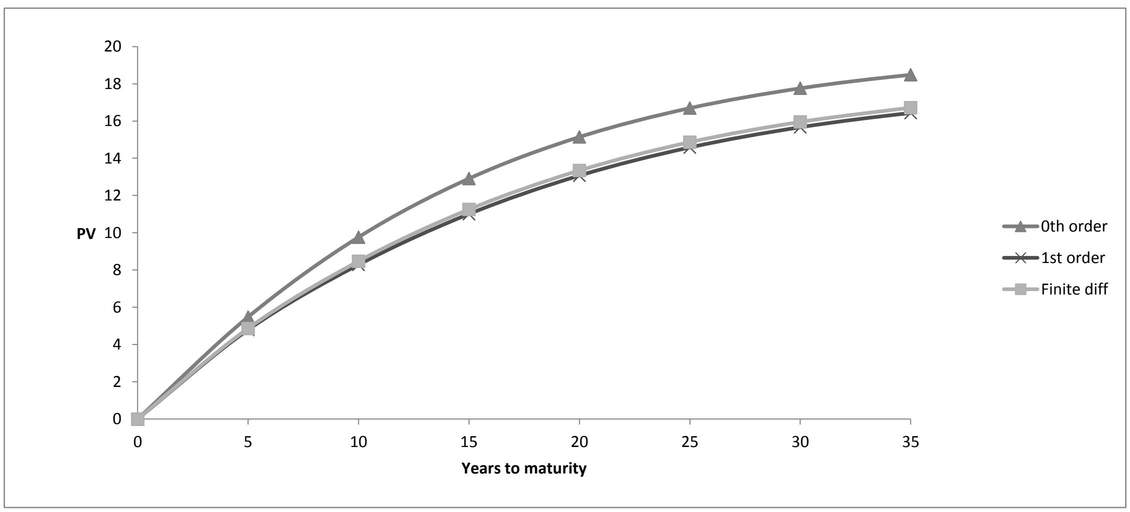

Figure 3: Finite difference calculation of equity recovery value vs. asymptotic expansion for various CoCo bond maturity dates.

TTM	0th order	1st order	Finite diff
5	5.48	4.79	4.86
10	9.76	8.28	8.47
15	12.91	11.01	11.25
20	15.15	13.07	13.34
25	16.70	14.58	14.86
30	17.76	15.67	15.95
35	18.48	16.43	16.72

Table 2: Zeroth and first order expansions versus finite difference computation for various times to maturity (TTM)

As can be seen the agreement is reasonably good for the full first-order expansion even out to fairly long maturities of 35y, the discrepancy in no case exceeding 30bp of notional. The zero-order expansion performs visibly less well. This is not surprising since all conversion intensity volatility effects (in particular correlation with equity) are excluded at this level of approximation.

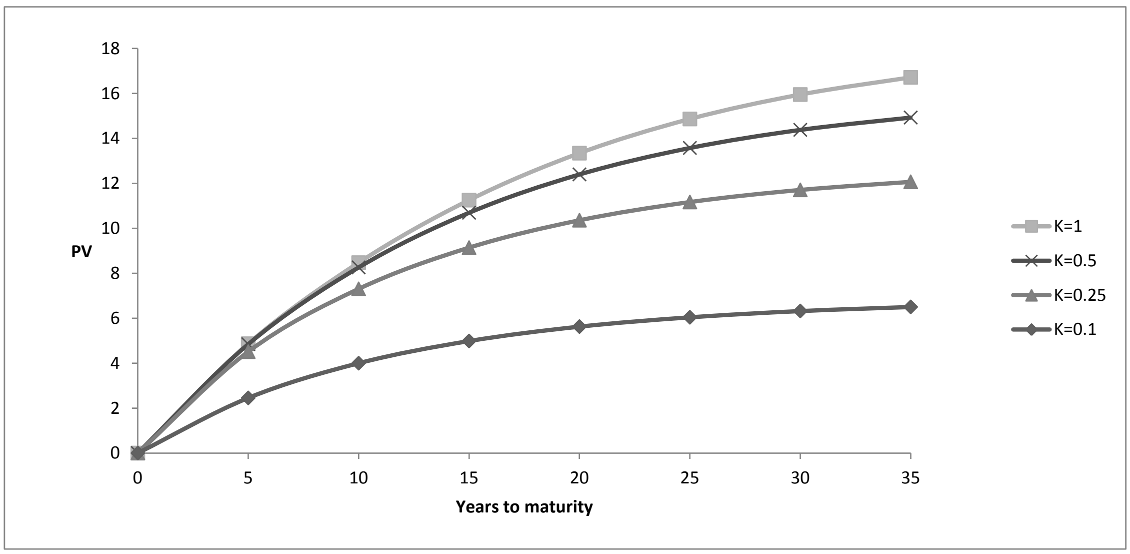

Figure 4: Equity recovery values for various CoCo bond maturity dates with different assumed values of K.

For reference, the impact of adjusting the upper limit K on the equity recovery payoff is illustrated in Fig. 4. Only the finite difference results are shown in this case. As can be seen, results for short maturity with K near 1 lose their dependence on the precise value of K, but results for smaller values of K or longer maturities demonstrate a more marked dependence, reflecting the fact that K is effectively the strike of a short call option embedded in the equity recovery payoff.

Comparisons are also made for the 15y bond[7] with different assumed levels of $\rho_{\lambda S}$ and of σ_λ. These are shown in Figs. 5 and 6, with the data presented in Tables 3 and 4 respectively. In both cases the first order expansion is seen to yield a linear dependence in line with the form of Eq. (27).

[7] A fairly long maturity bond was chosen for these comparisons because from Fig. 3 this was seen to be a more rigorous test of the validity of the asymptotic expansion than the shorter-term finite-maturity bonds (typically 10y) traded in the market.

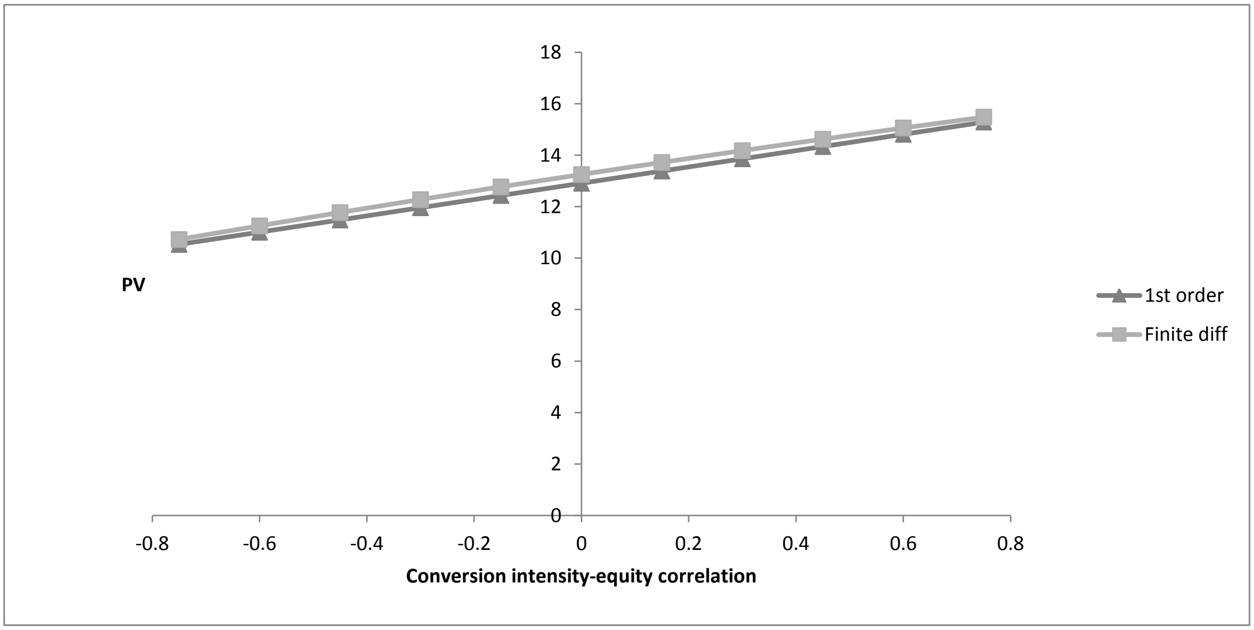

Figure 5: Finite difference calculation of equity recovery value vs. asymptotic expansion with various assumed conversion intensity-equity correlation levels.

$\rho_{\lambda S}$	1st order	Finite diff	Diff
-0.75	10.53	10.73	-0.19
-0.60	11.01	11.25	-0.25
-0.45	11.48	11.77	-0.29
-0.30	11.96	12.28	-0.32
-0.15	12.44	12.77	-0.33
0.00	12.91	13.25	-0.34
0.15	13.39	13.72	-0.33
0.30	13.86	14.18	-0.32
0.45	14.34	14.63	-0.29
0.60	14.82	15.06	-0.25
0.75	15.29	15.48	-0.19

Table 3: First order expansion versus finite difference computation for various correlation levels

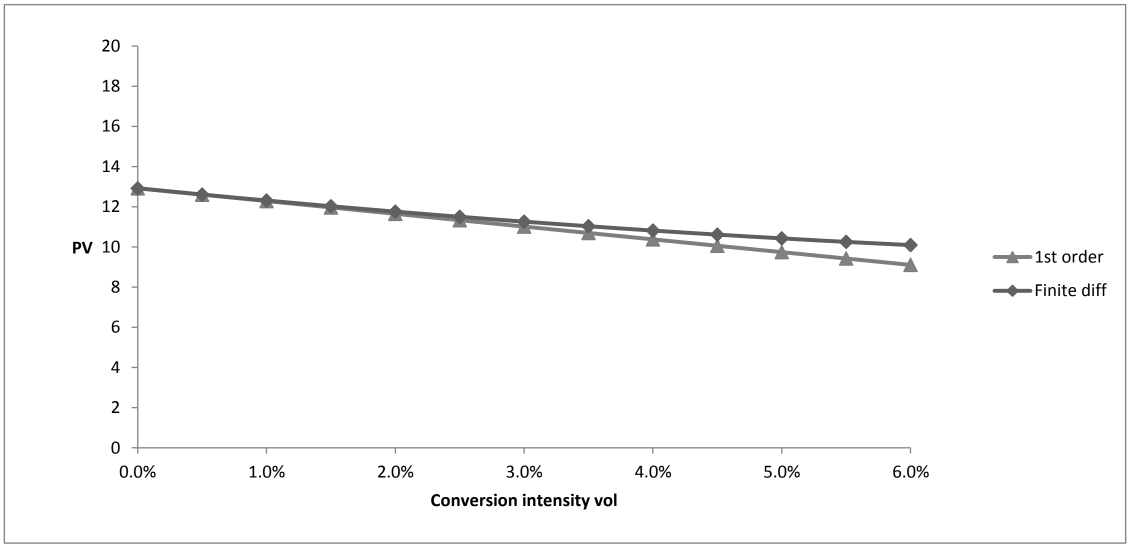

Figure 6: Finite difference calculation of equity recovery value vs. asymptotic expansion with various assumed normal volatility levels for equity conversion intensity.

Volatility	Finite diff	1st order	Diff
0.00%	12.91	12.91	0.00
0.50%	12.59	12.60	-0.01
1.00%	12.28	12.30	-0.03
1.50%	11.96	12.02	-0.06
2.00%	11.64	11.75	-0.11
2.50%	11.33	11.50	-0.17
3.00%	11.01	11.25	-0.25
3.50%	10.69	11.03	-0.34
4.00%	10.37	10.81	-0.44
4.50%	10.06	10.61	-0.55
5.00%	9.74	10.42	-0.68
5.50%	9.42	10.25	-0.83
6.00%	9.10	10.09	-0.99

Table 4: First order expansion versus finite difference computation for various conversion intensity volatility levels

The good agreement with the finite difference calculations in the first case supports the neglect of higher order terms in the expansion. However, as can be seen there is a progressive breakdown in the second case when the intensity volatility is pushed to higher levels. This is to be expected, since the accuracy of the expansion depends upon the smallness of the intensity volatility. We would suggest that values of σ_λ not exceeding about 3% are probably appropriate in practice, in which case the first order expansion should be usable. We note however that, for longer maturities, the accuracy is likely to diminish further, as can be inferred from the trend evident in Fig. 3 and Table 2.

We mentioned above the potential importance of our model in estimating equity delta. We illustrate the dependence of the equity recovery value of the 15y bond on equity spot price in Fig. 7 and Table 5.

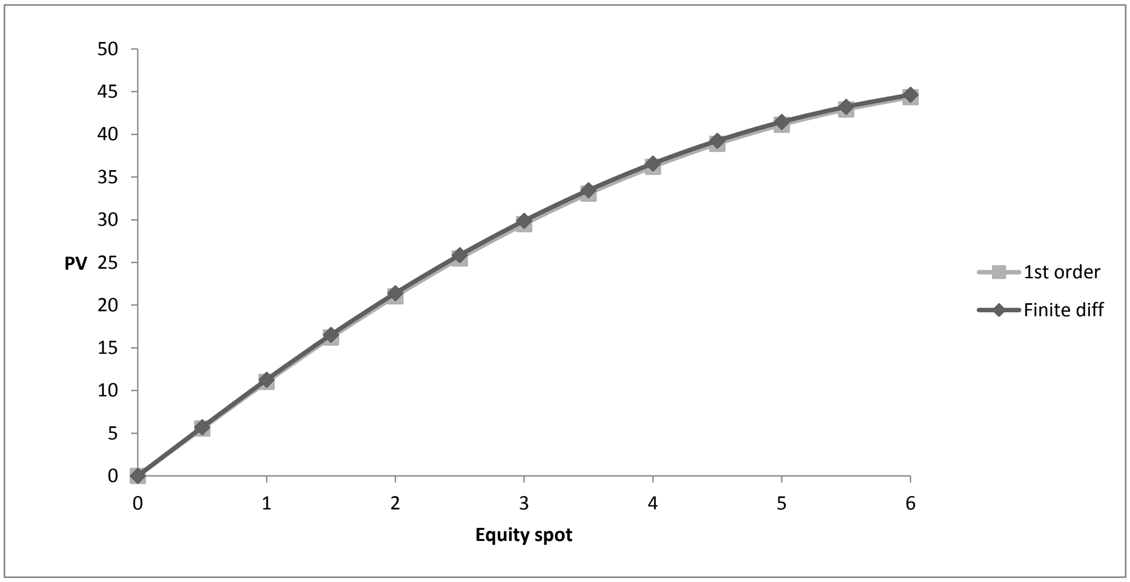

Figure 7: Finite difference calculation of equity recovery value vs. asymptotic expansion for different equity spot levels.

Equity spot	Finite diff	1st order	Diff
0.5	5.55	5.70	-0.15
1.0	11.01	11.25	-0.25
1.5	16.20	16.51	-0.31
2.0	21.03	21.39	-0.36
2.5	25.47	25.85	-0.38
3.0	29.48	29.87	-0.39
3.5	33.06	33.45	-0.39
4.0	36.20	36.57	-0.37
4.5	38.88	39.24	-0.35
5.0	41.12	41.45	-0.33
5.5	42.91	43.21	-0.31
6.0	44.35	44.63	-0.28

Table 5: First order expansion versus finite difference computation for various initial equity spot prices

As expected, the graph rises initially from zero but eventually starts to plateau off. The evident convexity of the curve shows the equity spot delta ($\partial \mathrm{PV}/\partial S_{t_0}$) to be a decreasing positive function of S_{t_0}. Indeed such monotonic behaviour can be inferred from the form of the leading order term of Eq. (27).

CoCo bond PV dependence on equity spot price is presented by De Spiegeleer and Schoutens (2012b) in their Fig. 2, for their equity derivatives model. There is qualitative similarity with our calculations for the equity recovery value. By contrast, however, their model predicts much less smooth behaviour for equity spot delta, with a spike for S_t around the postulated critical value S^* for short times to maturity; they explain this in terms of a phenomenon whereby CoCo bond holders would look to short equity to hedge this heightened delta, which phenomenon they term a "death spiral." On the basis of our model we would question the validity of this conclusion, both in terms of the non-monotonicity of the equity spot delta and of this behaviour being attributed in their model only to bonds with short times to maturity. We note also that when the views of the two models on the delta only of the *equity recovery* payment are compared, while our

model predicts an intuitively plausible positive value in all cases, the model of De Spiegeleer and Schoutens (2012b) would by its construction necessarily give rise to negative values, since the payoff amount is by assumption fixed while the probability of payment *increases* as the equity price approaches S^* from above. We would suggest, at the very least, that care should be taken in using model-predicted equity delta values so as to ensure they are properly representative of plausible market behaviour, not just of contingencies of the model.

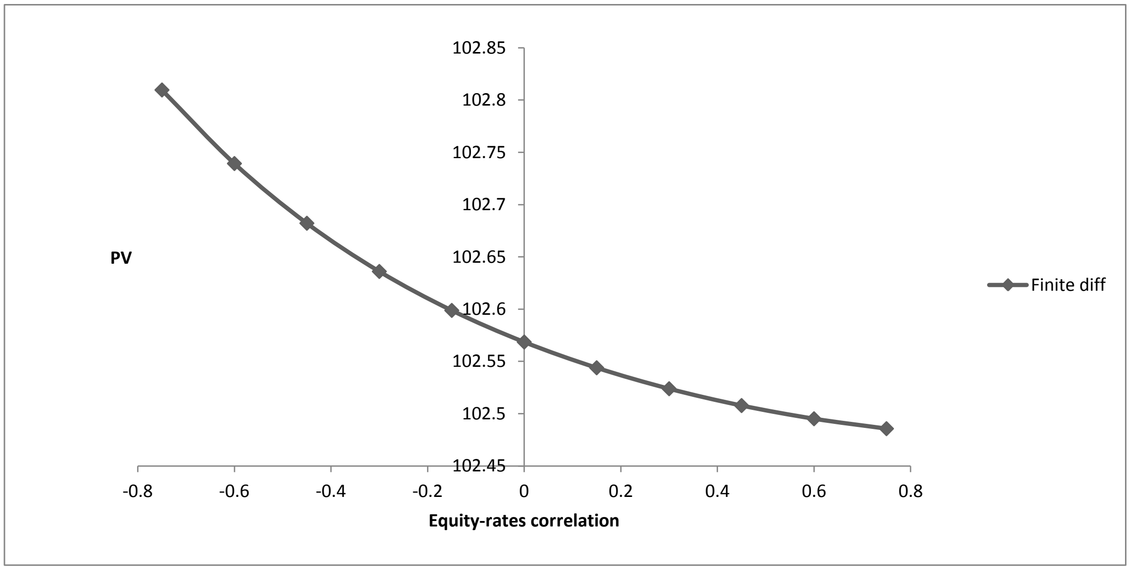

Figure 8: Impact of assumed equity-rates correlation level on bond PV for COCO_XS0459086822.

Finally we consider briefly the potential impact of stochastic rates on our results. We suppose a Black-Karasinski short rate model:

$$d(\ln r_t) = -a_r(\ln r_t - \theta_r(t))dt + \sigma_r dW_{r,t}, \tag{34}$$

with a constant local volatility level of $\sigma_r = 35\%$ and a mean reversion rate of $a_r = 0.25$. We allow that r_t can be correlated with both the equity and the equity conversion intensity processes. Figs. 8 and 9, calculated using our finite difference procedure, illustrate the impact of varying these correlations on CoCo bond XS0459086822. Since this bond has nearly ten years to run, it represents a fairly stringent test of our assumption that the impact of stochastic rates can be ignored.

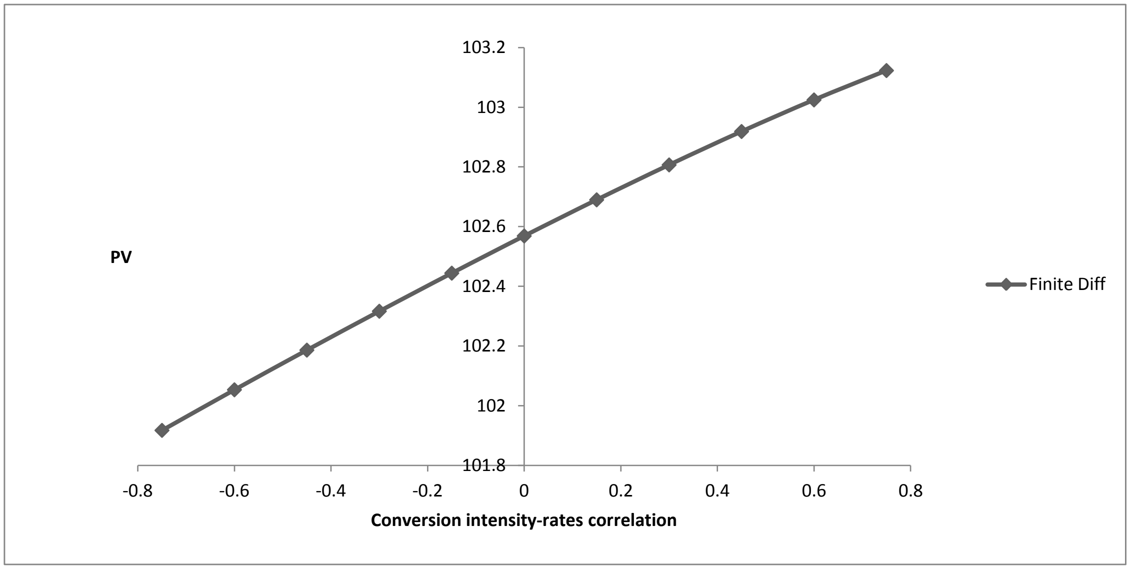

Figure 9: Impact of assumed conversion intensity-rates correlation level on bond PV for COCO_XS0459086822.

From Fig. 8, the impact of an equity-rates correlation shift of $\pm 50\%$ is seen to be at most 13 bp, which is less than the level of discrepancy already implicit in the use of our first order expansion. So neglect or inclusion of this correlation is not expected to have any significant impact on the quality of the modelling. By contrast, the conversion intensity-rates correlation is seen from Fig. 9 to have an impact about three times larger, which is arguably becoming potentially significant. However, it should be borne in mind that, since conversion intensity levels are not independently measurable from market data but must be inferred implicitly by fitting the model to bond prices, we have no practical means of inferring what is an appropriate correlation level. What is more, different assumed correlation levels would only result in a slight adjustment to the inferred conversion intensity level, with the impact of a $\pm 50\%$ correlation shift being found to be only of the order of 10bp on an assumed 850bp equity conversion intensity (see Table 1). Given the implicit uncertainty in the other aspects of CoCo bond modelling, this would have to be considered of minimal concern. We conclude on this basis that the established practice of ignoring stochastic rates in pricing CoCo bonds is justifiable.

6 Conclusion

We have developed a new equity-hybrid modelling approach for CoCo bond pricing which facilitates the calculation of delta risk with respect to both equity and conversion intensity; indeed it equally allows vega to be inferred. We find that, in order to calibrate our model to market data, a fairly large downward jump in the equity price ($\sim$80%) has to be assumed to occur at or around a conversion event.

We have further shown how the PDE derived can be solved in closed form under the assumption of a small equity conversion intensity volatility and presented an expression which is asymptotically accurate to first order. This is seen for (normal) volatilities up to around 3% in a Hull-White short rate model to give rise to PV estimates which are accurate to within 20-30 bp of notional for typical market circumstances. This is considered good enough to be used for most practical purposes.

It is possible to refine the asymptotic modelling to obtain more accurate results. Work is currently under way to look at the impact of second order terms. Also consideration has been given to performing an alternative analysis under the assumption that the intensity rather than its volatility is small. This would potentially allow larger volatility levels to be employed without compromising accuracy.

A Proof of Theorem 3.1

We prove Theorem 3.1 solving Eq. (23) in the standard manner by successive levels of approximation. At zeroth order we must solve

$$\mathcal{L}[h_0(x,t)] = -\overline{\lambda}(t)\min\{M_0(x,t),K\}$$

for $t \in D_m$ with final condition that $h_0(x,T)=0$. This can be achieved by means of the following readily obtainable Green's function for the canonical diffusion operator $\mathcal{L}[.]$:

$$G(\mathbf{z},t;\boldsymbol{\zeta},v) = B(t,v)H(v-t)\phi(\mathbf{z};\boldsymbol{\zeta},R(t,v)) \tag{35}$$

where $\mathbf{z}=(x,y)^T$, $\boldsymbol{\zeta}=(\xi,\eta)^T$, $B(t,v)$ is the risky discount factor defined in Eq. (8) above, $H(.)$ is the Heaviside step function and $\phi(\mathbf{z};\boldsymbol{\zeta},R(t,v))$ is a two-dimensional Gaussian probability distribution function with mean $\boldsymbol{\zeta}$ and covariance matrix

$$R(t,v) = \begin{pmatrix} I_S(t,v) & I_\rho(t,v) \\ I_\rho(t,v) & I_y(t,v) \end{pmatrix} \tag{36}$$

where

$$I_\rho(t_1,t_2) = \rho_{\lambda S}\int_{t_1}^{t_2}\sigma_y(u)\sigma_S(u)du,$$

and $I_S(t_1,t_2)$ and $I_y(t_1,t_2)$ are defined in Eqs. (20) and (25). Making use of Eq. (35) we obtain

$$\begin{aligned} h_0(x,t) &= \int_t^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{z},t;\boldsymbol{\zeta},v)\overline{\lambda}(v)\min\{M_0(\xi,v),K\}\,d\xi\,d\eta\,dv \\ &= \int_t^T \overline{\lambda}(v)B(t,v) \\ &\quad \Big(e^{x-\frac{1}{2}I_S(t_0,t)}M(1+k)F(v)N(-d_1(x,t,v)) + KN(d_2(x,t,v))\Big)dv \end{aligned} \tag{37}$$

where $d_1(.,.,.)$, $d_2(.,.,.)$ and $N(.)$ are defined in Eqs. (29)-(31) above. Continuing to first order we find[8]

$$\mathcal{L}[h_1(x,y,t)] = ye^{-a(t-t_0)}\left(-\min\{M_0(x,t),K\} + h_0 + k\frac{\partial h_0}{\partial x}\right)$$

for $t \in D_m$ with final condition that $h_1(x,y,T)=0$. The solution can be written

$$h_1(x,y,t) = h_{1,0}(x,t) + yh_{1,1}(x,t),$$

where, applying our Green's function and integrating over η, we find

$$\begin{aligned} h_{1,0}(x,t) &= \frac{\partial}{\partial x}\int_t^T\int_{-\infty}^{\infty} B(t,v)N'\left(\frac{\xi-x}{\sqrt{I_S(t,v)}}\right)e^{-a(v-t_0)}I_\rho(t,v)\cdot \\ &\qquad (\min\{M_0(\xi,v),K\} - h_0(\xi,v) - k\frac{\partial h_0(\xi,v)}{\partial \xi})\,d\xi\,dv \\ &= e^{x-\frac{1}{2}I_S(t_0,t)}M(1+k)\int_t^T F(v)B(t,v)e^{-a(v-t_0)}I_\rho(t,v)N(-d_1(x,t,v))dv \\ &\quad - e^{x-\frac{1}{2}I_S(t_0,t)}M(1+k)\int_t^T F(v)B(t,v)\overline{\lambda}(v)\int_t^v e^{-a(u-t_0)}I_\rho(t,u)\cdot \\ &\qquad \left(1+k+k\frac{\partial}{\partial x}\right)N(-d_1(x,t,v))du\,dv. \end{aligned} \tag{38}$$

[8] We note that, given the fact that at leading order there is no y-dependence, the last term in Eq. (23) is in fact $O(\epsilon^2)$ so can be ignored here.

Likewise

$$
\begin{aligned}
h_{1,1}(x,t) &= \int_t^T \int_{-\infty}^{\infty} B(t,v) N'\left(\frac{\xi - x}{\sqrt{I_S(t,v)}}\right) e^{-a(v-t_0)} \left(\min\{M_0(\xi,v),K\} - h_0(\xi,v) - k\frac{\partial h_0(\xi,v)}{\partial \xi}\right) d\xi\, dv \\
&= \int_t^T B(t,v) e^{-a(v-t_0)} \left(e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)F(v)N(-d_1(x,t,v)) + KN(d_2(x,t,v))\right) dv \\
&\quad - \int_t^T B(t,v)\overline{\lambda}(v) \int_t^v e^{-a(u-t_0)} \\
&\qquad \left(e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)^2 F(v) N(-d_1(x,t,v)) + KN(d_2(x,t,v))\right) du\, dv.
\end{aligned}
\tag{39}
$$

Substituting the above expressions into Eq. (26), setting $t = t_0$ and reverting to unscaled notation, we obtain Eq. (27).

It remains to prove that the error in using Eq. (27) in place of an exact expression for $f_{\text{conv}}(t_0)$ is indeed $O(\epsilon^2)$. Let us start by writing our first order approximate solution to Eq. (23) in scaled notation as

$$
h^{(\epsilon)}(x,y,t) := h_0(x,t) + \epsilon h_1(x,y,t)
$$

Further write the full operator $\hat{\mathcal{L}}[.]$ defined in Eq. (18) in *scaled* notation as

$$
\mathcal{L}^*[.] := \mathcal{L}[.] - \epsilon e^{-a(t-t_0)} \left(y + ky\frac{\partial}{\partial x} - I_y(t_0,t)\frac{\partial}{\partial y}\right)
$$

and define the difference between the exact and approximate solution as

$$
\Delta h^{(\epsilon)}(x,y,t) = h(x,y,t) - h^{(\epsilon)}(x,y,t).
$$

We immediately see that this satisfies

$$
\begin{aligned}
\mathcal{L}^*[\Delta h^{(\epsilon)}(x,y,t)] &:= \epsilon^2 y e^{-a(t-t_0)} \left(1 + k\frac{\partial}{\partial x}\right) h_{1,0}(x,t) \\
&\quad + \epsilon^2 e^{-a(t-t_0))} \left(y^2 - I_y(t_0,t) + y^2 k \frac{\partial}{\partial x}\right) h_{1,1}(x,t)
\end{aligned}
$$

for $t \in D_m$ with final condition that $\Delta h^{(\epsilon)}(x,y,T) = 0$. Furthermore, as the difference between two uniformly bounded functions, $\Delta h^{(\epsilon)}(x,y,t)$ is uniformly bounded as $T \to \infty$. Since the solution to the corresponding homogeneous equation is null, it is immediately apparent that the solution to the nonhomogeneous equation is $O(\epsilon^2)$ on its domain of definition and hence in particular at $t = t_0$.

This concludes the proof of the theorem. □

B Proof of Corollary 3.1

Consider first Eq. (13). Subject to our condition (a), the contribution to $f_{\text{surv}}(t_0)$ from terms with $i > i^*$ will be bounded as $T \to \infty$ by the (convergent) geometric series

$$
c\Delta_M B_0(t_{i^*}) \sum_{i=i^*+1}^{\infty} e^{-\zeta_m \Delta_m}
$$

where $\Delta_M := \sup_i\{\Delta t_i\}$ and $\Delta_m := \inf_i\{\Delta t_i\}$. The principle of dominated convergence ensures the convergence of $f_{\text{surv}}(t_0)$ in this case. Subject to our condition (b) there will be a real constant $F^* > 0$

such that $F_i \leq F^* \ \forall i > i^*$. We infer therefrom by an argument similar to the previous that $f_{\mathrm{surv}}(t_0)$ as defined in Eq. (14) will also be convergent.

Finally we demonstrate that the approximation for $f_{\mathrm{conv}}(t_0)$ in Eq. (27) remains valid as $T \to \infty$. This follows immediately from the observation made in Theorem 3.1 and its proof that both $h(x, y, t)$ and its approximation $h^{(\epsilon)}(x, y, t)$ are uniformly bounded as $T \to \infty$ and their difference is $O(\epsilon^2)$. □

Acknowledgements

The authors are indebted to Dr A. Jacquier of Imperial College, London for numerous helpful comments in relation to an earlier draft of this paper.

References

Alòs, E. (2006): 'A generalization of the Hull and White formula with applications to option pricing approximation', Finance and Stochastics Vol. 10, No. 3, pp. 353-365.

Antonelli, F., A. Ramponi, S. Scarlatti (2010) 'Exchange Option Pricing under Stochastic Volatility: a Correlation Expansion', Review of Derivatives Research, Vol. 13, pp. 45-73.

Antonelli, F., S. Scarlatti (2009) 'Pricing options under stochastic volatility: a power series approach', Finance and Stochastics, vol. 13, pp. 269-303.

Ayache, E., P.A. Forsyth, K.R. Vetzal (2002): 'Next generation Models for Convertible Bonds with Credit Risk', Wilmott Magazine December, pp. 68-77.

Benhamou, E., E. Gobet, M. Miri (2010), 'Expansion formulas for European options in a local volatility model', International Journal of Theoretical and Applied Finance, Vol. 13, No. 4, pp. 603-634.

Brigo, D., J. Garcia, and N. Pede 'CoCo Bonds Valuation with Equity- and Credit-Calibrated First Passage Structural Models' International Journal of Theoretical and Applied Finance, vol. 18, No. 4, (2015), pp. 219–249.

Brigo, D., F. Mercurio (2006): 'Interest Rate Models—Theory and Practice: With Smile, Inflation and Credit', 2nd ed., Springer Finance.

Byström, H. (2005): 'Credit Default Swaps and Equity Prices: The Itraxx CDS Index Market.' Working Papers, Department of Economics, Lund University; No. 24.

De Spiegeleer, J., W. Schoutens (2012a): 'Pricing Contingent Convertibles: A Derivatives Approach', The Journal of Derivatives, Vol. 20, No. 2, pp. 27-36.

De Spiegeleer, J., W. Schoutens (2012b): 'Steering a bank around a death spiral: Multiple Trigger CoCos', Wilmott Magazine, Vol. 59, pp. 62-69.

Ehlers, P., P. Schönbucher (2004): 'The Influence of FX Risk on Credit Spreads', Working Paper, http://www.actuaries.org/AFIR/Colloquia/Zurich/Ehlers_Schoenburcher.pdf.

EL-Mohammadi, R. (2009): 'BSWithJump Model And Pricing Of Quanto CDS With FX Devaluation Risk', MPRA Paper No. 42781, https://mpra.ub.uni-muenchen.de/42781/.

Fouque, J.-P., G. Papanicolaou, R. Sircar, K. Sølna (2011): 'Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives' (Cambridge University Press), 456pp.

Gobet, E., J. Hok (2014): 'Expansion Formulas for Bivariate Payoffs with Application to Best-of Options on Equity and Inflation', International Journal of Theoretical and Applied Finance, Vol. 17, No. 2, pp. 219-249.

Hull, J., A. White (1990:) 'Pricing Interest Rate Derivative Securities', The Review of Financial Studies, Vol. 3, pp. 573-592.

Kim, Y., N. Kunitomo (1999): 'Pricing Options under Stochastic Interest Rates: A New Approach', Asia-Pacific Financial Markets, Vol. 6, No. 1, pp. 49-70.

Li, D.X. (2000): 'On Default Correlation: A Copula Approach', Journal of Fixed Income, Vol. 9, pp. 43-54.

Muroi, Y. (2005): 'Pricing contingent claims with credit risk: Asymptotic expansion approach', Finance and Stochastics, Vol. 9, No. 3, pp. 415-427.

Muroi, Y., E.K. Takino (2010): 'Pricing Derivatives using the Asymptotic Expansion Approach: Credit Migration Models with Stochastic Credit Spreads', Asia-Pacific Financial Markets, Vol. 18, No. 4, pp. 345-372.

Park, J., Y. Lee, J. Lee (2013): 'Pricing of Quanto Option under the Hull and White Stochastic Volatility Model', Communications of the Korean Mathematical Society, Vol. 28, No. 3, pp. 615-633.

Pennacchi, G. (2010): 'A structural model of contingent bank capital', Federal Reserve Bank of Cleveland Working Paper, `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1595080`.

Serjantov, A. (2011): 'New Techniques for Modelling Hybrid Capital Securities', paper presented at symposium, Global Derivatives Trading and Risk Management, April, 2011.

Shiraya, K., A. Takahashi (2011): 'Pricing Average Options on Commodities', Journal of Futures Markets, Vol. 31, No. 5, pp. 407-439.

Shiraya, K., A. Takahashi (2015): 'Pricing Multi-Asset Cross Currency Options', Journal of Futures Markets, Vol. 34, No. 1, pp. 1-19.

Shiraya, K., A. Takahashi, M. Toda (2011): 'Pricing Barrier and Average Options under Stochastic Volatility Environment', Journal of Computational Finance, Vol. 15, No. 2, pp. 111-148.

Shiraya, K., A. Takahashi, A. Yamazaki (2012): 'Pricing Swaptions under the LIBOR Market Model of Interest Rates with Local-Stochastic Volatility Models', Wilmott Magazine, Vol. 61, pp.48-63.

Wilkens, S., N. Bethke (2014): 'Contingent Convertible (CoCo) Bonds: A First Empirical Assessment of Selected Pricing Models', Financial Analysts Journal, Vol. 70, No. 2, pp. 59-75.